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BIVARIATE C^1 QUADRATIC FINITE ELEMENTS AND VERTEX SPLINES

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ABSTRACT. Following work of Heindl and of Powell and Sabin, each triangle of an arbitrary (regular) triangulation Δ of a polygonal region Ω in $\mathbb{R}^2$ is subdivided into twelve triangles, using the three medians, yielding the refinement $\widehat{\Delta}$ of Δ, so that C^1 quadratic finite elements can be constructed. In this paper, we derive the Bézier nets of these elements in terms of the parameters that describe function and first partial derivative values at the vertices and values of the normal derivatives at the midpoints of the edges of Δ. Consequently, bivariate C^1 quadratic (generalized) vertex splines on Δ have an explicit formulation. Here, a generalized vertex spline is one which is a piecewise polynomial on the refined grid partition $\widehat{\Delta}$ and has support that contains at most one vertex of the original partition Δ in its interior. The collection of all C^1 quadratic generalized vertex splines on Δ so constructed is shown to form a basis of $S_2^1(\widehat{\Delta})$, the vector space of all functions on $C^1(\Omega)$ whose restrictions to each triangular cell of the partition $\widehat{\Delta}$ are quadratic polynomials. A subspace with the basis given by appropriately chosen generalized vertex splines with exactly one vertex of Δ in the interior of their supports, that reproduces all quadratic polynomials, is identified, and hence, has approximation order three. Quasi-interpolation formulas using this subspace are obtained. In addition, a constructive procedure that yields a locally supported basis of yet another subspace with dimension given by the number of vertices of Δ, that has approximation order three, is given.

1. INTRODUCTION

Let Ω be a simply connected region in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a simple closed polygonal Jordan curve. Also, let Δ be a (regular) triangulation of Ω, and by this, we mean that the complement of Δ relative to Ω consists of a finite number of triangles such that none of the vertices of any triangle lies on the edge of another triangle. For $-1 \le r < d$, where r and d are integers, $S_d^r(\Delta)$ will denote the vector space of all functions in $C^r(\Omega)$ whose restrictions on each triangular region of the partition Δ are polynomials of total degree at most d. The space $S_d^r(\Delta)$ is called a bivariate spline space. We are interested

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in functions belonging to $S_d^r(\Delta)$ with smallest supports. If d is allowed to be fairly large as compared with r, then a basis of $S_d^r(\Delta)$ consisting of functions whose supports contain at most one vertex of Δ in the interior can be constructed. Such basis elements are called vertex splines and were introduced by Chui and Lai [3]. For instance, for $r = 1$, d must be at least 4 (cf. [1, 3]). Since lower-degree piecewise polynomials are more desirable from the practical point of view, such as computational efficiency and capability in the control of geometric characteristics, we consider construction of macroelements by refining the grid partition Δ. For $r = 1$, if we wish to use the smallest degree d, which is 2, we may subdivide each triangle into 12 triangles by using the three medians as shown in Figure 1 (cf. Heindl [7] and Powell and Sabin [9]). The refinement of Δ so obtained will be denoted by $\hat{\Delta}$. We will now consider the spline space $S_2^1(\hat{\Delta})$. A function in $S_2^1(\hat{\Delta})$ whose support contains at most one vertex of the original partition Δ in its interior will be called a *generalized vertex spline.* In Figures 2(a) to 2(d), we demonstrate the supports of all possible generalized vertex splines, where the dotted lines denote the refinement of Δ that defines $\hat{\Delta}$.

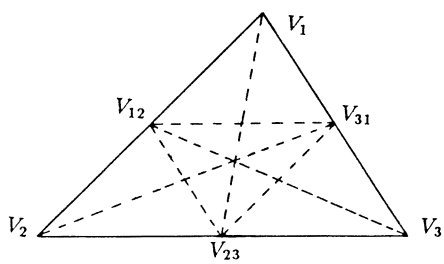

FIGURE 1

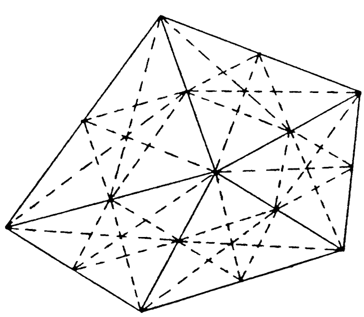

FIGURE 2(a)

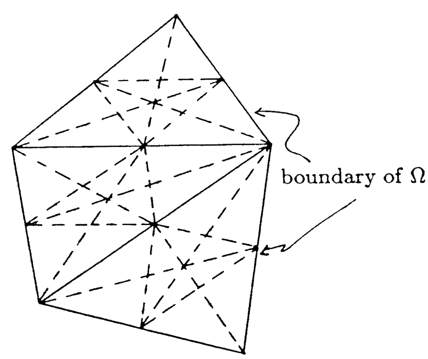

FIGURE 2(b)

We will first give an explicit formulation of all generalized vertex splines in $S_2^1(\hat{\Delta})$ by displaying the Bézier nets of the polynomial pieces and show that they form a basis of $S_2^1(\hat{\Delta})$. Then we will derive a necessary and sufficient condition

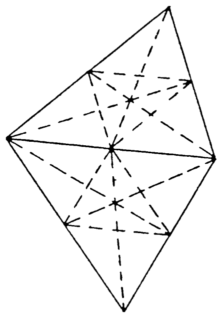

FIGURE 2(c)

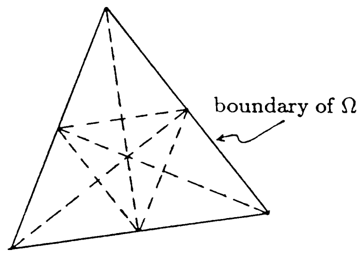

FIGURE 2(d)

on the generalized vertex spline basis of a subspace $\widehat{S}_2^1(\widehat{\Delta})$ of $S_2^1(\widehat{\Delta})$, obtained by deleting basis elements of $S_2^1(\widehat{\Delta})$ that have supports given by Figures 2(c) and 2(d), so that $\widehat{S}_2^1(\widehat{\Delta})$ still contains π_2^2, the space of all quadratic polynomials in two variables. Consequently, the approximation order of $\widehat{S}_2^1(\widehat{\Delta})$ remains 3. An important advantage of the subspace $\widehat{S}_2^1(\widehat{\Delta})$ over the whole space $S_2^1(\widehat{\Delta})$ is that it is easier to construct a quasi-interpolation formula, which requires only function values at the vertices of the original triangulation Δ, if the generalized vertex splines with supports given by Figures 2(c) and 2(d) are not being used. This topic will also be studied in this paper. Finally, we study a constructive procedure to determine yet another subspace of dimension given by the number of vertices of Δ, that still maintains third-order approximation.

2. CONSTRUCTION OF MACROELEMENTS

We will first study the construction of a so-called C^1 quadratic macroelement. Let T be a triangular region with vertices $V_i = (a_i, b_i)$, $i = 1, 2, 3$. We are interested in constructing a C^1 piecewise quadratic polynomial on T with "boundary values" so chosen that when another macroelement on an adjacent triangle sharing a common edge with T is constructed with the same boundary conditions on this common edge, a C^1 function on the union of these two triangles is obtained. We will give the Bézier net of each polynomial piece of the macroelement. Following Powell and Sabin [9] and Heindl [7], we divide T into twelve triangles as shown in Figure 1, using the medians V_1V_{23}, V_2V_{31}, and V_3V_{12}, as well as the line segments $V_{12}V_{23}$, $V_{23}V_{31}$, and $V_{31}V_{12}$, where

$$V_{12} = \tfrac{1}{2}(V_1 + V_2), \quad V_{23} = \tfrac{1}{2}(V_2 + V_3), \quad V_{31} = \tfrac{1}{2}(V_3 + V_1). \tag{2.1}$$

We will use the parameters $d_1, d_2, d_3, m_1, m_2, m_3, n_1, n_2, n_3$, and p_1, p_2, p_3, where for $i = 1, 2, 3$, d_i denotes the function value at V_i, m_i and n_i denote values of the first partial derivatives with respect to x and y, respectively, at V_i, and

$$\begin{cases} p_1 = D_{\mathbf{k}_1} f(V_{12}), \\ p_2 = D_{\mathbf{k}_2} f(V_{23}), \\ p_3 = D_{\mathbf{k}_3} f(V_{31}), \end{cases} \tag{2.2}$$

with $\mathbf{k}_1 = (b_1 - b_2, a_2 - a_1)$, $\mathbf{k}_2 = (b_2 - b_3, a_3 - a_2)$, and $\mathbf{k}_3 = (b_3 - b_1, a_1 - a_3)$. Here and throughout, for any vector $\mathbf{k} = (k_1, k_2)$, $D_{\mathbf{k}} f$ denotes the derivative of f with respect to $\mathbf{k}$ defined by

$$D_{\mathbf{k}} f = \mathbf{k} \cdot \nabla f = k_1 \frac{\partial f}{\partial x} + k_2 \frac{\partial f}{\partial y}.$$

Hence, p_1, p_2, and p_3 are "normal derivatives" at the midpoints of the edges V_1V_2, V_2V_3, and V_3V_1 of T, respectively.

Since Bézier nets will be determined, we must use the barycentric coordinates relative to each of the twelve subtriangles. Let T_1 and T_2 be two of these twelve triangles adjacent to each other, as shown in Figure 3, where we have used X_1, X_2, X_3, and X_4 to denote the vertices. The barycentric coordinates relative to T_1 and T_2 will be denoted by (u, v, w) and $(\tilde{u}, \tilde{v}, \tilde{w})$, respectively; that is, for each $X = (x, y)$ in $\mathbb{R}^2$, we have

$$X = uX_1 + vX_2 + wX_3 = \tilde{u}X_1 + \tilde{v}X_4 + \tilde{w}X_2,$$

where u, v, w, $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ are linear polynomials in (x, y), with $u + v + w = \tilde{u} + \tilde{v} + \tilde{w} = 1$. For any two quadratic polynomials p and q defined on T_1 and T_2, respectively, we may write

$$p(u, v, w) = \sum_{i+j+k=2} a_{ijk} \frac{2!}{i!j!k!} u^i v^j w^k \tag{2.3}$$

and

$$q(\tilde{u}, \tilde{v}, \tilde{w}) = \sum_{i+j+k=2} b_{ijk} \frac{2!}{i!j!k!} \tilde{u}^i \tilde{v}^j \tilde{w}^k. \tag{2.4}$$

Then $\{a_{ijk}\}$ and $\{b_{ijk}\}$ are called the Bézier nets of p and q on the triangles T_1 and T_2, respectively. These coefficient values can be displayed on the corresponding triangles as shown in Figure 3, and define the corresponding polynomials uniquely. Let f be a function whose restrictions on T_1 and T_2 are p and q, respectively. The following smoothness condition on f will be useful in constructing the macroelement (cf. Farin [5]).

Lemma 2.1. *The function f is in $C^1(T_1 \cup T_2)$ if and only if*

$$b_{200} = a_{200}, \quad b_{101} = a_{110}, \quad b_{002} = a_{020},$$
$$b_{110} = u^0 a_{200} + v^0 a_{110} + w^0 a_{101},$$

and

$$b_{011} = u^0 a_{110} + v^0 a_{020} + w^0 a_{011},$$

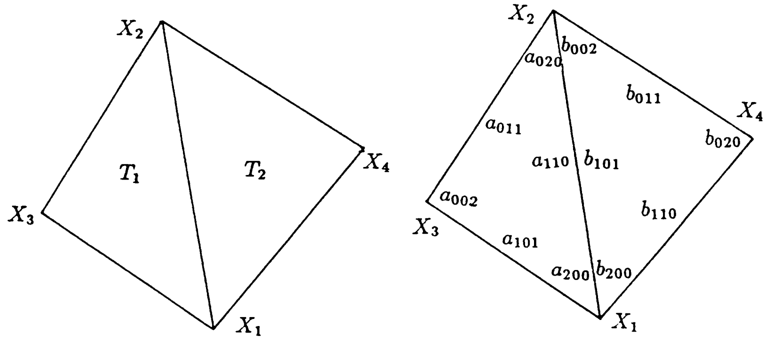

FIGURE 3

where (u^0, v^0, w^0) *is the barycentric coordinate of* X_4 *relative to the triangle* T_1, *in the sense that*

$$X_4 = u^0 X_1 + v^0 X_2 + w^0 X_3 .$$

Let us now apply this lemma to construct piecewise polynomial functions on the triangle T with vertices V_1, V_2, V_3 as shown in Figure 1. At one of the six corner triangles where V_1, V_2, or V_3 is one of the vertices, we will utilize the parameters d_i, m_i, n_i, $i = 1, 2, 3$. Let X_1 denote V_1, V_2, or V_3, and X_2, X_3, X_4 the other appropriate vertices V_{12}, V_{23}, V_{31} as shown in one of the three situations in Figure 4. Then the following result, which will be useful for our construction procedure and can be easily verified, relates the Bézier net to values of the function and its derivatives.

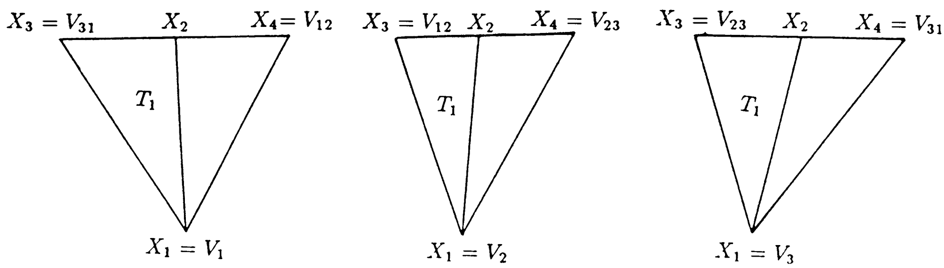

FIGURE 4

Lemma 2.2. *Let* $\{a_{ijk}\}$, $i + j + k = 2$, *be the Bézier net of the quadratic polynomial* $p(u, v, w)$ *in* (2.3) *on triangle* T_1 *as shown in Figures* 3 *and* 4. *Then* $a_{200} = p(X_1)$,

$$a_{110} = p(X_1) + \tfrac{1}{2} D_{X_2 - X_1} p(X_1),$$

and

$$a_{101} = p(X_1) + \tfrac{1}{2} D_{X_3 - X_1} p(X_1).$$

Let us now return to the given triangular region T which is subdivided into twelve triangles as shown in Figure 1. Since the triangulation is a crosscut partition with six crosscuts and four interior vertices, it follows that the dimension of this bivariate C^1 piecewise quadratic spline space is

$$\binom{2+2}{2} + 6\binom{(2-1)+1}{2} + \sum_{i=1}^{4} 0 = 12,$$

since for $r = 1$ and $d = 2$ the contribution from each of the four vertices is zero (cf. Chui and Wang [4]). This fact has already been observed in Powell and Sabin [9], where it is also shown that the twelve parameters d_i, m_i, n_i, and p_i, $i = 1, 2, 3$, indeed determine the space uniquely. Hence, we may now proceed to construct the Bézier nets of our macroelement f on T in terms of these parameters.

An advantage in using the medians to triangulate T is that the point X_2 in each of the three situations in Figure 4 is the midpoint of the line segment X_3X_4, so that the barycentric coordinate (u^0, v^0, w^0) of X_4 relative to the triangle T_1 in Lemma 2.1 is given by

$$(u^0, v^0, w^0) = (0, 2, -1). \tag{2.5}$$

Hence, the Bézier nets of the three pairs of corner subtriangles in Figure 4, as well as those of the corresponding three pairs of inner triangles, can be somewhat simplified. In fact, it is clear that they can be expressed in the form shown in Figure 5. The objective of this section is to express all the Bézier nets, shown in Figure 5, of the C^1 piecewise quadratic macroelement f in terms of the twelve parameters:

$$d_i = f(V_i), \quad m_i = \frac{\partial}{\partial x} f(V_i), \quad n_i = \frac{\partial}{\partial y} f(V_i) \tag{2.6}$$

and p_i as defined in (2.2), where $i = 1, 2, 3$. First, let the Bézier nets of the twelve polynomials be expressed in terms of the parameters α_{ij}, β_{ij}, γ_{ij}, and μ_{ij}, where Lemma 2.1 has been used to relate some of them. The others will be expressed in terms of d_i, m_i, n_i, and p_i.

It is clear that $\mu_{ii} = d_i$ for $i = 1, 2, 3$. By applying Lemma 2.2 to the corner triangles, we may also express the γ_{ij} values in terms of $d_1, d_2, d_3, m_1, m_2, m_3, n_1, n_2, n_3$. In order to apply Lemma 2.2 to the inner triangles, we must first derive function values and values of the first partial derivates at V_{12}, V_{23}, and V_{31}, which are vertices of the inner triangles. We have the following result.

Lemma 2.3. *For* $i, j = 1, 2, 3$,

$$\mu_{ij} = \frac{d_i + d_j}{2} + \frac{1}{8}[(m_i - m_j)(a_j - a_i) + (n_i - n_j)(b_j - b_i)] \tag{2.7}$$

and

$$\begin{cases} D_{V_1 - V_{12}} f(V_{12}) = (d_1 - d_2) - \dfrac{m_1 + m_2}{4}(a_1 - a_2) - \dfrac{n_1 + n_2}{4}(b_1 - b_2), \\ D_{V_2 - V_{23}} f(V_{23}) = (d_2 - d_3) - \dfrac{m_2 + m_3}{4}(a_2 - a_3) - \dfrac{n_2 + n_3}{4}(b_2 - b_3), \\ D_{V_3 - V_{31}} f(V_{31}) = (d_3 - d_1) - \dfrac{m_3 + m_1}{4}(a_3 - a_1) - \dfrac{n_3 + n_1}{4}(b_3 - b_1). \end{cases} \tag{2.8}$$

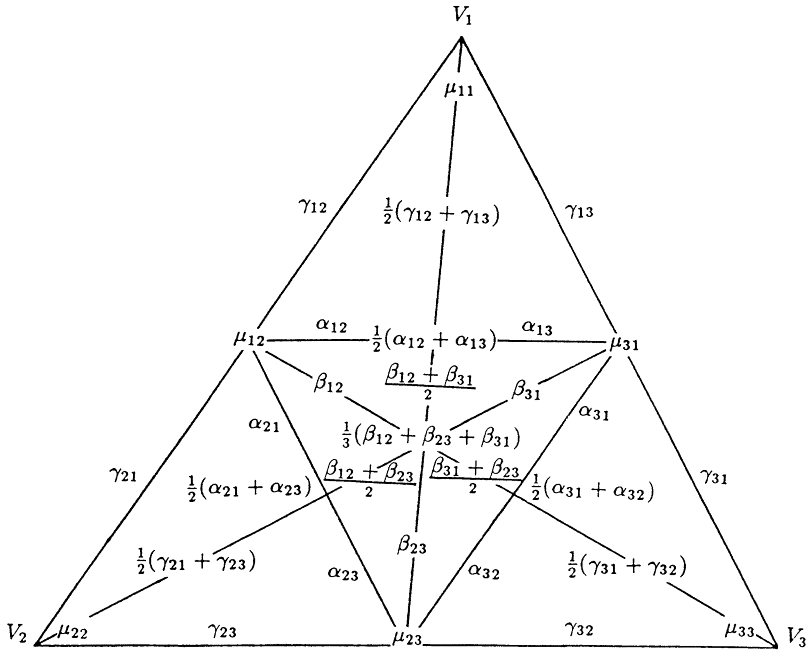

FIGURE 5

Proof. Since the restriction of f to each of the three edges of the original triangle T is a univariate quadratic spline, the function and derivative values of this spline function at the knot V_{ij} can be easily expressed in terms of its function and derivative values at the two adjacent knots V_i and V_j. This gives (2.7) and (2.8), respectively. □

In order to be able to apply Lemma 2.2 to the inner triangles, we need the partial derivatives of f with respect to x and y at V_{ij}. These values can be obtained by solving the linear systems

$$(2.9)\qquad \begin{cases} (a_1 - a_2)\frac{\partial}{\partial x}f(V_{12}) + (b_1 - b_2)\frac{\partial}{\partial y}f(V_{12}) = 2D_{V_1 - V_{12}}f(V_{12}), \\ (b_1 - b_2)\frac{\partial}{\partial x}f(V_{12}) - (a_1 - a_2)\frac{\partial}{\partial y}f(V_{12}) = p_1, \end{cases}$$

$$(2.10)\qquad \begin{cases} (a_2 - a_3)\frac{\partial}{\partial x}f(V_{23}) + (b_2 - b_3)\frac{\partial}{\partial y}f(V_{23}) = 2D_{V_2 - V_{23}}f(V_{23}), \\ (b_2 - b_3)\frac{\partial}{\partial x}f(V_{23}) - (a_2 - a_3)\frac{\partial}{\partial y}f(V_{23}) = p_2, \end{cases}$$

and

$$(2.11)\qquad \begin{cases} (a_3 - a_1)\frac{\partial}{\partial x}f(V_{31}) + (b_3 - b_1)\frac{\partial}{\partial y}f(V_{31}) = 2D_{V_3 - V_{31}}f(V_{31}), \\ (b_3 - b_1)\frac{\partial}{\partial x}f(V_{31}) - (a_3 - a_1)\frac{\partial}{\partial y}f(V_{31}) = p_3, \end{cases}$$

where the values on the right of the first equations of the linear systems are given by (2.8) in Lemma 2.3, and the second equations of the linear systems are simply (2.2). Hence, by using the values of $f(V_{ij}) = \mu_{ij}$ given by (2.7) and the values of $\frac{\partial}{\partial x}f(V_{ij})$ and $\frac{\partial}{\partial y}f(V_{ij})$ which can be obtained by solving (2.9)–(2.11), we may apply Lemma 2.2 to derive the values of α_{ij} and β_{ij} shown in Figure 5. We summarize these results in the following theorem.

Theorem 2.1. *Let T be a triangular region with vertices $V_1 = (a_1, b_1)$, $V_2 = (a_2, b_2)$, and $V_3 = (a_3, b_3)$, and let $f \in C^1(T)$ whose restrictions to the twelve triangles shown in Figure 1 are bivariate quadratic polynomials. Then the Bézier nets of these polynomials, shown in Figure 5 and uniquely determined by the parameters d_i, m_i, n_i, and p_i, $i = 1, 2, 3$, defined by* (2.6) *and* (2.2), *are given by*

$$\mu_{ij} = \frac{d_i + d_j}{2} + \frac{1}{8}[(m_i - m_j)(a_j - a_i) + (n_i - n_j)(b_j - b_i)],$$

$$\gamma_{ij} = d_i + \frac{1}{4}[m_i(a_j - a_i) + n_i(b_j - b_i)],$$

$$\alpha_{ij} = \mu_{ij} + \frac{1}{8}\left[(a_k - a_j)\frac{\partial}{\partial x}f(V_{ij}) + (b_k - b_j)\frac{\partial}{\partial y}f(V_{ij})\right],$$

and

$$\beta_{ij} = \mu_{ij} + \frac{1}{24}\left[\left(a_k - \frac{a_i + a_j}{2}\right)\frac{\partial}{\partial x}f(V_{ij}) + \left(b_k - \frac{b_i + b_j}{2}\right)\frac{\partial}{\partial y}f(V_{ij})\right],$$

where $i, j = 1, 2, 3$, k is the complement of $\{i, j\}$ relative to $\{1, 2, 3\}$ whenever $i \neq j$, and the first partial derivatives of f at V_{ij} are determined by (2.9)–(2.11) *with $D_{V_k - V_{ij}}f(V_{ij})$ given by* (2.8).

3. Generalized vertex splines

Let Ω be a bounded simply connected polygonal region in $\mathbb{R}^2$ and Δ a regular but otherwise arbitrary triangulation of Ω. We denote the vertices and edges of Δ by $V_1 = (a_1, b_1), \ldots, V_\eta = (a_\eta, b_\eta)$ and $e_1, \ldots, e_\xi$, respectively, where both interior and boundary vertices and edges are enumerated. (See Figure 6 where $\eta = 15$ and $\xi = 31$.)

Subdivide each triangle in Δ into twelve triangles by using the midpoints of the edges $e_1, \ldots, e_\xi$ as shown in Figure 1, yielding a triangulation $\widehat{\Delta}$ which is a refinement of the partition Δ of Ω. (See Figure 7.) We are interested in studying the bivariate C^1 quadratic spline space $S_2^1(\widehat{\Delta})$ of functions in $C^1(\Omega)$ whose restrictions to each triangular subregion with respect to the triangulation $\widehat{\Delta}$ are functions π_2^2, the collection of bivariate polynomials of total degree at most 2. For each $j = 1, \ldots, \xi$, let W_j denote the midpoint of the edge e_j of the original triangulation Δ. Consider an arbitrary spline function f in $S_2^1(\widehat{\Delta})$. Let p_j denote the "normal derivative" of f at W_j in a direction normal to e_j. For consistency, we may pick the direction to be

$$\mathbf{k}_j = (\tilde{b}_p^j - \tilde{b}_q^j, \tilde{a}_q^j - \tilde{a}_p^j) \tag{3.1}$$

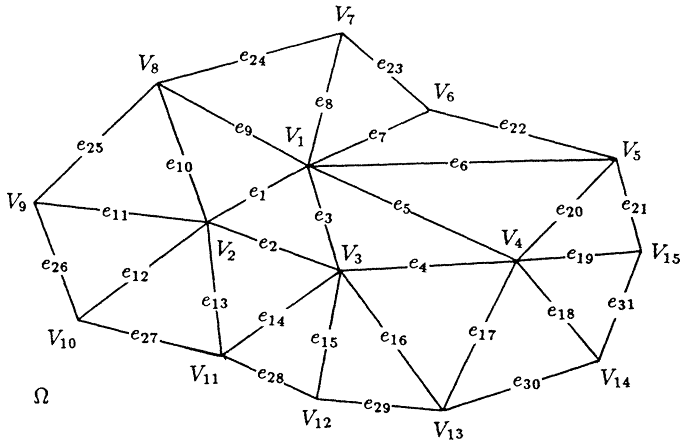

FIGURE 6. Triangulation Δ

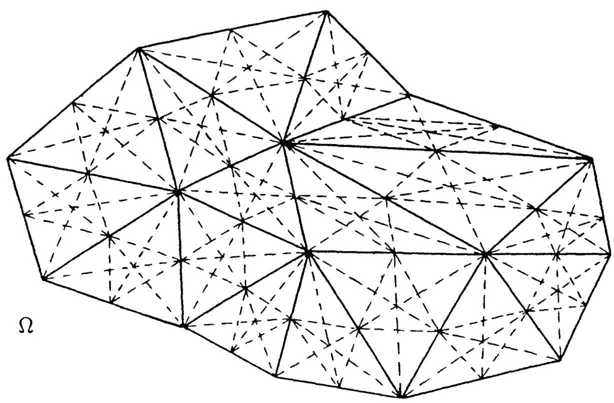

FIGURE 7. Triangulation $\widehat{\Delta}$

with $p < q$, where $V_p = (\tilde{a}_p^j, \tilde{b}_p^j)$ and $V_q = (\tilde{a}_q^j, \tilde{b}_q^j)$ are the endpoints of the edge e_j. In addition, let

$$d_i = f(V_i), \quad m_i = \frac{\partial}{\partial x} f(V_i), \quad n_i = \frac{\partial}{\partial y} f(V_i),$$

$i = 1, \dots, \eta$. Since the values d_i, m_i, and n_i, $i = 1, \dots, \eta$, and p_j, $j = 1, \dots, \xi$, uniquely determine f on each triangle of the partition Δ of Ω, and hence on all of Ω, it follows that the dimension of the spline space $S_2^1(\widehat{\Delta})$ is

$$\dim S_2^1(\widehat{\Delta}) = 3\eta + \xi. \tag{3.2}$$

In fact, a basis

$$\{S_i, T_i, U_i, E_j : i = 1, \dots, \eta, \; j = 1, \dots, \xi\} \tag{3.3}$$

of $S_2^1(\widehat{\Delta})$ is obtained by defining

$$(3.4)\quad \begin{cases} \left(S_i(V_k),\, \frac{\partial}{\partial x}S_i(V_k),\, \frac{\partial}{\partial y}S_i(V_k)\right) = (\delta_{ik}, 0, 0), & i, k = 1, \dots, \eta, \\ D_{\mathbf{k}_j} S_i(W_j) = 0, \quad i = 1, \dots, \eta \text{ and } j = 1, \dots, \xi, \end{cases}$$

$$(3.5)\quad \begin{cases} \left(T_i(V_k),\, \frac{\partial}{\partial x}T_i(V_k),\, \frac{\partial}{\partial y}T_i(V_k)\right) = (0, \delta_{ik}, 0), & i, k = 1, \dots, \eta, \\ D_{\mathbf{k}_j} T_i(W_j) = 0, \quad i = 1, \dots, \eta \text{ and } j = 1, \dots, \xi, \end{cases}$$

$$(3.6)\quad \begin{cases} \left(U_i(V_k),\, \frac{\partial}{\partial x}U_i(V_k),\, \frac{\partial}{\partial y}U_i(V_k)\right) = (0, 0, \delta_{ik}), & i, k = 1, \dots, \eta, \\ D_{\mathbf{k}_j} U_i(W_j) = 0, \quad i = 1, \dots, \eta \text{ and } j = 1, \dots, \xi, \end{cases}$$

and

$$(3.7)\quad \begin{cases} \left(E_j(V_k),\, \dfrac{\partial}{\partial x}E_j(V_k),\, \dfrac{\partial}{\partial y}E_j(V_k)\right) = (0, 0, 0), & k = 1, \dots, \eta \\ & \text{and } j = 1, \dots, \xi, \\ D_{\mathbf{k}_l} E_j(W_l) = \delta_{jl}, \quad j, l = 1, \dots, \xi. \end{cases}$$

Here, δ_{ik} is the Kronecker delta and $\mathbf{k}_j$ is defined in (3.1). It is clear that the collection (3.3) is a linearly independent set in $S_2^1(\widehat{\Delta})$. Hence, by (3.2), it is a basis of $S_2^1(\widehat{\Delta})$.

Another important observation is that the functions in (3.3) have "smallest" supports in the sense that each E_j has minimal support, and if $f \in S_2^1(\widehat{\Delta})$ has support properly contained in $\operatorname{supp} S_i$, $\operatorname{supp} T_i$, or $\operatorname{supp} U_i$, then f must be in the span of $\{E_j : j = 1, \dots, \eta\}$. Note that if V_i is an interior vertex of Δ, then $\operatorname{supp} S_i$, $\operatorname{supp} T_i$, and $\operatorname{supp} U_i$ are given by Figure 2(a), where V_i is the vertex interior to this support, and if V_i is a boundary vertex of Δ, then $\operatorname{supp} S_i$, $\operatorname{supp} T_i$, and $\operatorname{supp} U_i$ are given by Figure 2(b). Similarly, if e_j is an interior edge of Δ, then $\operatorname{supp} E_j$ is shown in Figure 2(c) with e_j being the edge interior to this support, and if e_j is a boundary edge of Δ, then $\operatorname{supp} E_j$ is shown in Figure 2(d). Since these supports, considered as supports of splines in $S_d^r(\Delta)$, where Δ is the original triangulation, are supports of the vertex splines in $S_d^r(\Delta)$ (cf. Chui and Lai [3]), we will call S_i, T_i, U_i, E_j *generalized vertex splines.* Hence, we have obtained the following result.

Proposition 3.1. *Let Δ be an arbitrary (regular) triangulation with η vertices and ξ edges, where both interior and boundary vertices and edges are counted, and let $\widehat{\Delta}$ be the refinement of Δ by subdividing each triangle of Δ into twelve triangles using the midpoints of the edges of Δ as shown in Figure 1. Then the bivariate spline space $S_2^1(\widehat{\Delta})$ has dimension $3\eta+\xi$ and a basis of $S_2^1(\widehat{\Delta})$ is given by the collection (3.3) of generalized vertex splines.*

In applications, however, the values of the "normal derivatives" at the midpoints W_j of the edges e_j, $j = 1, \dots, \xi$, are unknown quantities. Hence, the generalized vertex splines $E_1, \dots, E_\xi$ seem to be useless. In order to take full advantage of these functions, we incorporate them with the other basis functions S_i, T_i, and U_i. For each $i = 1, \dots, \eta$, let $e_1^i, \dots, e_{n_i}^i$ be all the edges e_j with V_i as the common vertex, ordered in the counterclockwise direction around V_i. Let $E_1^i, \dots, E_{n_i}^i$ be the corresponding generalized vertex splines among the collection $\{\pm E_j\}$, where the positive sign is chosen if the normal $\mathbf{k}_j$ points in the counterclockwise direction around V_i and the negative sign is chosen otherwise. For any n_i-tuples $\alpha^i = (\alpha_1^i, \dots, \alpha_{n_i}^i)$, $\beta^i = (\beta_1^i, \dots, \beta_{n_i}^i)$, and $\gamma^i = (\gamma_1^i, \dots, \gamma_{n_i}^i)$ in $\mathbb{R}^{n_i}$, set

$$
(3.8) \qquad \begin{cases} S_{\alpha^i} = S_i + \sum_{l=1}^{n_i} \alpha_l^i E_l^i, \\ T_{\beta^i} = T_i + \sum_{l=1}^{n_i} \beta_l^i E_l^i, \\ U_{\gamma^i} = U_i + \sum_{l=1}^{n_i} \gamma_l^i E_l^i, \end{cases}
$$

and consider the subspace $\widetilde{S}_2^1(\widehat{\Delta}; \alpha^i, \beta^i, \gamma^i)$ with basis

$$
(3.9) \qquad \{S_{\alpha^i}, T_{\beta^i}, U_{\gamma^i} : i = 1, \dots, \eta\}.
$$

In what follows, we will characterize the values of α^i, β^i, γ^i so that the subspace $\widetilde{S}_2^1(\widehat{\Delta}; \alpha^i, \beta^i, \gamma^i)$ contains π_2^2, the collection of all quadratic polynomials in $\mathbb{R}^2$.This subspace has the important property that it has the same third-order approximation as the entire space $S_2^1(\widehat{\Delta})$. To facilitate the discussion, we need the following notation.

Let $V_l^i = (a_l^i, b_l^i)$ denote the other endpoint of the edge e_l^i, $l = 1, \dots, n_i$ and $i = 1, \dots, \eta$. Consider $\widehat{\alpha}^i = (\widehat{\alpha}_1^i, \dots, \widehat{\alpha}_{n_i}^i)$, $\widehat{\beta}^i = (\widehat{\beta}_1^i, \dots, \widehat{\beta}_{n_i}^i)$, and $\widehat{\gamma}^i = (\widehat{\gamma}_1^i, \dots, \widehat{\gamma}_{n_i}^i)$, where, for $l = 1, \dots, n_i$,

$$
(3.10) \qquad \widehat{\alpha}_l^i = 0, \qquad \widehat{\beta}_l^i = \frac{1}{2|e_l^i|}(b_i - b_l^i), \quad \widehat{\gamma}_l^i = \frac{1}{2|e_l^i|}(a_l^i - a_i).
$$

Here, $|e_l^i|$ denotes the length of the edge e_l^i. Now, following the suggestion of Heindl [7] and Powell and Sabin [9], we set

$$
(3.11) \qquad \begin{cases} S_i^* = S_i + \sum_{l=1}^{n_i} \widehat{\alpha}_l^i E_l^i = S_i, \\ T_i^* = T_i + \sum_{l=1}^{n_i} \widehat{\beta}_l^i E_l^i, \\ U_i^* = U_i + \sum_{l=1}^{n_i} \widehat{\gamma}_l^i E_l^i. \end{cases}
$$

An example of these functions can be found in our report [2]. Let $\widehat{S}_2^1(\widehat{\Delta}) := \widetilde{S}_2^1(\widehat{\Delta}; \widehat{\alpha}^i, \widehat{\beta}^i, \widehat{\gamma}^i)$ be the subspace with basis $\{S_i^*, T_i^*, U_i^* : i = 1, \dots, \eta\}$. The main result in this section is that this is the "unique" subspace that contains all of π_2^2.

Theorem 3.1. *We have $\pi_2^2 \subset \widetilde{S}_2^1(\widehat{\Delta}; \alpha^i, \beta^i, \gamma^i)$ if and only if $\widetilde{S}_2^1(\widehat{\Delta}; \alpha^i, \beta^i, \gamma^i) = \widehat{S}_2^1(\widehat{\Delta})$, or equivalently $\alpha^i = \widehat{\alpha}^i$, $\beta^i = \widehat{\beta}^i$, and $\gamma^i = \widehat{\gamma}^i$.*

Proof. (i) *Necessity.* Assume that $\pi_2^2 \subset \widetilde{S}_2^1(\widehat{\Delta})$. Then by (3.4), (3.5), (3.6), and (3.8), we have, for any $p \in \pi_2^2$,

$$p = \sum_i p(V_i)S_i + \sum_i \frac{\partial p}{\partial x}(V_i)T_i + \sum_i \frac{\partial p}{\partial y}(V_i)U_i + \sum_i \sum_{l=1}^{n_i} \left[\alpha_l^i p(V_i) + \beta_l^i \frac{\partial p}{\partial x}(V_i) + \gamma_l^i \frac{\partial p}{\partial y}(V_i)\right] E_l^i. \tag{3.12}$$

Set $W_l^i = (V_i + V_l^i)/2$,and $\mathbf{k}_l^i = (b_i - b_l^i, a_l^i - a_i)/|e_l^i|$. Then it follows from (3.4)–(3.7) and (3.12) that

$$D_{\mathbf{k}_l^i} p(W_l^i) = \alpha_l^i[p(V_i) + p(V_l^i)] + \beta_l^i \left[\frac{\partial p}{\partial x}(V_i) + \frac{\partial p}{\partial x}(V_l^i)\right] + \gamma_l^i \left[\frac{\partial p}{\partial y}(V_i) + \frac{\partial p}{\partial y}(V_l^i)\right]. \tag{3.13}$$

On the other hand, it is clear that

$$\begin{aligned} D_{\mathbf{k}_l^i} p(W_l^i) &= \frac{1}{2}[D_{\mathbf{k}_l^i} p(V_i) + D_{\mathbf{k}_l^i} p(V_l^i)] \\ &= \frac{1}{2}\left[\left(\frac{\partial p}{\partial x}(V_i) + \frac{\partial p}{\partial x}(V_l^i)\right)(b_i - b_l^i)/|e_l^i| + \left(\frac{\partial p}{\partial y}(V_i) + \frac{\partial p}{\partial y}(V_l^i)\right)(a_l^i - a_i)/|e_l^i|\right]. \end{aligned} \tag{3.14}$$

Hence, by using (3.10), we have

$$(\alpha_l^i - \widehat{\alpha}_l^i)[p(V_i) + p(V_l^i)] + (\beta_l^i - \widehat{\beta}_l^i)\left[\frac{\partial p}{\partial x}(V_i) + \frac{\partial p}{\partial x}(V_l^i)\right] + (\gamma_l^i - \widehat{\gamma}_l^i)\left[\frac{\partial p}{\partial y}(V_i) + \frac{\partial p}{\partial y}(V_l^i)\right] = 0$$

for all $l = 1, \ldots, n_i$ and $i = 1, \ldots, \eta$. By choosing $p(x, y) = 1,\ x, y$, consecutively, we obtain $\alpha_l^i = \widehat{\alpha}_l^i$, $\beta_l^i = \widehat{\beta}_l^i$, and $\gamma_l^i = \widehat{\gamma}_l^i$.

(ii) *Sufficiency.* Consider the "Hermite operator" H from $C^1(\Omega)$ to $\widehat{S}_2^1(\widehat{\Delta})$ defined by

$$H(f) := \sum_{i=1}^{\eta} \left[f(V_i)S_i^* + \frac{\partial f}{\partial x}(V_i)T_i^* + \frac{\partial f}{\partial y}(V_i)U_i^*\right]. \tag{3.15}$$

From (3.11), we have

$$H(f) = \sum_{i=1}^{\eta} \left\{f(V_i)S_i + \frac{\partial f}{\partial x}(V_i)T_i + \frac{\partial f}{\partial y}(V_i)U_i + \sum_{l=1}^{n_i}\left[\widehat{\beta}_l^i \frac{\partial f}{\partial x}(V_i) + \widehat{\gamma}_l^i \frac{\partial f}{\partial y}(V_i)\right] E_l^i\right\}. \tag{3.16}$$

It is sufficient to prove that $H(p) = p$ for all $p \in \pi_2^2$. To do so, we may apply Proposition 3.1 and the Hermite properties (3.4)–(3.7) to write

$$p = \sum_{i=1}^{\eta} \left\{ p(V_i)S_i + \frac{\partial p}{\partial x}(V_i)T_i + \frac{\partial p}{\partial y}(V_i)U_i \right\} + \sum_{j=1}^{\xi} D_{\mathbf{k}_j} p(W_j)E_j, \tag{3.17}$$

where the normal vectors $\mathbf{k}_j$ are defined in (3.1). For each edge e_j of the original partition Δ, we denote its endpoints by the vertices

$$V_i = (a_i, b_i) \quad \text{and} \quad V_l^i = (a_l^i, b_l^i)$$

or by the vertices

$$V_t = (a_t, b_t) \quad \text{and} \quad V_m^t = (a_m^t, b_m^t),$$

where $V_i = V_m^t$ and $V_t = V_l^i$. Hence, the normal vector to the edge e_j is either

$$\mathbf{k}_j = \mathbf{k}_l^i = -\mathbf{k}_m^t \quad \text{or} \quad \mathbf{k}_j = -\mathbf{k}_l^i = \mathbf{k}_m^t.$$

Without loss of generality, we assume that $\mathbf{k}_j = \mathbf{k}_l^i = -\mathbf{k}_m^t$, where

$$\mathbf{k}_l^i = (b_i - b_l^i, a_l^i - a_i)/|e_l^i| \tag{3.18}$$

and

$$-\mathbf{k}_m^t = (b_m^t - b_t, a_t - a_m^t)/|e_m^t| \tag{3.19}$$

with $|e_l^i| = |e_m^t| = |e_j|$. Hence,

$$E_j = E_l^i = -E_m^t. \tag{3.20}$$

Since $\mathbf{k}_j = (\mathbf{k}_l^i - \mathbf{k}_m^t)/2$ and $\frac{\partial p}{\partial x}(x, y)$ and $\frac{\partial p}{\partial y}(x, y)$ are both linear polynomials, so that their values at $W_j = W_l^i = W_m^t$ are the averages at the two endpoints $V_i = V_m^t$ and $V_t = V_l^i$, we have, by using (3.10), (3.18), (3.19), and (3.20),

$$\begin{aligned} D_{\mathbf{k}_j} p(W_j)E_j &= \frac{1}{2}\{D_{\mathbf{k}_l^i} p(W_l^i) - D_{\mathbf{k}_m^t} p(W_m^t)\}E_j \\ &= \frac{1}{2}\left\{ \widehat{\beta}_l^i \left[\frac{\partial p}{\partial x}(V_i) + \frac{\partial p}{\partial x}(V_l^i)\right] + \widehat{\gamma}_l^i \left[\frac{\partial p}{\partial y}(V_i) + \frac{\partial p}{\partial y}(V_l^i)\right] \right. \\ &\qquad \left. - \widehat{\beta}_m^t \left[\frac{\partial p}{\partial x}(V_t) + \frac{\partial p}{\partial x}(V_m^t)\right] - \widehat{\gamma}_m^t \left[\frac{\partial p}{\partial y}(V_t) + \frac{\partial p}{\partial y}(V_m^t)\right] \right\} E_j \\ &= \left\{ \widehat{\beta}_l^i \frac{\partial p}{\partial x}(V_i) + \widehat{\gamma}_l^i \frac{\partial p}{\partial y}(V_i) \right\} E_l^i + \left\{ \widehat{\beta}_m^t \frac{\partial p}{\partial x}(V_t) + \widehat{\gamma}_m^t \frac{\partial p}{\partial y}(V_t) \right\} E_m^t. \end{aligned}$$

Hence, by putting this quantity into (3.17), we have

$$\begin{aligned} p = \sum_{i=1}^{\eta} \Bigg\{ & p(V_i)S_i + \frac{\partial p}{\partial x}(V_i)T_i + \frac{\partial p}{\partial y}(V_i)U_i \\ & + \sum_{l=1}^{n_i} \left[\widehat{\beta}_l^i \frac{\partial p}{\partial x}(V_i) + \widehat{\gamma}_l^i \frac{\partial p}{\partial y}(V_i) \right] E_l^i \Bigg\}, \end{aligned}$$

or, by (3.16), $H(p) = p$ for all $p \in \pi_2^2$. □

We remark that since the Hermite operator H defined in (3.15) preserves all quadratic polynomials, its approximation order is full. More precisely, let $H_2^r(\Omega)$ denote the Sobolev space with norm

$$\|f\|_{r,\Omega} := \left(\sum_{|\alpha| \le r} \|D^\alpha f\|_2^2 \right)^{1/2},$$

where $\| \, \|_2$ is the L^2 norm on Ω and

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial x^{\alpha_2}} f,$$

$\alpha = (\alpha_1, \alpha_2)$. Also, denote by $\delta = |\Delta|$ the maximum of the diameters of the triangles in Δ. Then we have the following result (cf. [8]).

Theorem 3.2. *There exists a positive constant* C *such that*

$$\|Hf - f\|_{r,\Omega} \le C\delta^{3-r} \|f\|_{3,r}$$

for all $f \in H_2^3(\Omega)$ *and* $r = 0, \ldots, 3$.

For $r = 0$, we can even apply the Sobolev Imbedding Theorem to obtain the following inequality:

$$\|Hf - f\|_\infty \le C\delta^3 \|f\|_{3,\Omega} \tag{3.21}$$

for all $f \in H_2^3(\Omega)$, where the L^∞ norm on Ω is used.

4. Quasi-interpolation by generalized vertex splines

We are now ready to study the construction of approximation formulas by using generalized vertex splines in $S_2^1(\widehat{\Delta})$ so that the optimal order of approximation $O(\delta^3)$ is attained. Since only function values at the vertices of the original grid partition Δ will be used, the location of these vertices plays an important role in the construction scheme. For simplicity, we will assume the existence of a positive constant c such that the $c\delta$-neighborhood

$$\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - V_i| < c\delta\}$$

of each V_i contains at least five other vertices, say $V_1^i, \ldots, V_5^i$, of Δ such that these six vertices (including V_i itself) do not lie on a quadratic algebraic curve. Here, of course, the union of two straight lines is considered to be such a curve. Hence, there is a unique polynomial

$$p_2(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + g \tag{4.1}$$

that interpolates any function f at these six points. In other words, the linear system

$$\begin{cases} p_2(V_i) = f(V_i), \\ p_2(V_l^i) = f(V_l^i), \qquad l = 1, \ldots, 5, \end{cases} \tag{4.2}$$

has a unique solution. In order to obtain a quasi-interpolation formula that involves only function values, we must express the derivatives

$$Z_1^i := \frac{\partial p_2}{\partial x}(V_i) \quad \text{and} \quad Z_2^i := \frac{\partial p_2}{\partial y}(V_i)$$

in terms of the values in (4.2). From (4.2) we can obtain the following system of equations with unknowns Z_1^i, Z_2^i, a, b, and c:

(4.3)
$$Z_1^i(a_l^i - a_i) + Z_2^i(b_l^i - b_i) + a(a_l^i - a_i)^2 + 2b(a_l^i - a_i)(b_l^i - b_i) + c(b_l^i - b_i)^2 = f(V_l^i) - f(V_i), \quad l = 1, \dots, 5.$$

Hence, by using Cramer's Formula, we have

(4.4)
$$Z_1^i = \det \begin{bmatrix} f(V_1^i) - f(V_i) & (b_1^i - b_i) & (a_1^i - a_i)^2 & 2(a_1^i - a_i)(b_1^i - b_i) & (b_1^i - b_i)^2 \\ f(V_2^i) - f(V_i) & (b_2^i - b_i) & (a_2^i - a_i)^2 & 2(a_2^i - a_i)(b_2^i - b_i) & (b_2^i - b_i)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f(V_5^i) - f(V_i) & (b_5^i - b_i) & (a_5^i - a_i)^2 & 2(a_5^i - a_i)(b_5^i - b_i) & (b_5^i - b_i)^2 \end{bmatrix} \Big/ Z,$$

(4.5)
$$Z_2^i = \det \begin{bmatrix} a_1^i - a_i & f(V_1^i) - f(V_i) & (a_1^i - a_i)^2 & 2(a_1^i - a_i)(b_1^i - b_i) & (b_1^i - b_i)^2 \\ a_2^i - a_i & f(V_2^i) - f(V_i) & (a_2^i - a_i)^2 & 2(a_2^i - a_i)(b_2^i - b_i) & (b_2^i - b_i)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_5^i - a_i & f(V_5^i) - f(V_i) & (a_5^i - a_i)^2 & 2(a_5^i - a_i)(b_5^i - b_i) & (b_5^i - b_i)^2 \end{bmatrix} \Big/ Z,$$

where

$$Z = \det \begin{bmatrix} a_1^i - a_i & b_1^i - b_i & (a_1^i - a_i)^2 & 2(a_1^i - a_i)(b_1^i - b_i) & (b_1^i - b_i)^2 \\ a_2^i - a_i & b_2^i - b_i & (a_2^i - a_i)^2 & 2(a_2^i - a_i)(b_2^i - b_i) & (b_2^i - b_i)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_5^i - a_i & b_5^i - b_i & (a_5^i - a_i)^2 & 2(a_5^i - a_i)(b_5^i - b_i) & (b_5^i - b_i)^2 \end{bmatrix}. \tag{4.6}$$

That is, we arrive at the following "quasi-interpolation" formula:

$$S(f) = \sum_i f(V_i)S_i^* + \sum_i Z_1^i T_i^* + \sum_i Z_2^i V_i^* \tag{4.7}$$

which clearly satisfies $S(p) \equiv p$ for all $p \in \pi_2^2$. In other words, the approximation order of the spline operator $S(f)$ is 3, which is the full approximation order from the space $S_2^1(\widehat{\Delta})$. One important feature of this approximation formula is that it can be formulated as

$$S(f) = \sum_i f(V_i)B_i^*, \tag{4.8}$$

where each spline function B_i^* has compact support. In fact, explicit formulations of B_i^* can be obtained. In order to simplify the construction procedure, let us study the situation where two pairs of the vertices V_l^i are colinear with V_i. Under this assumption, we only need four V_l^i's (instead of five) in determining Z_1^i and Z_2^i. For example, let $\{V_i, V_1^i, V_3^i\}$ and $\{V_i, V_2^i, V_4^i\}$ lie on two straight lines intersecting at V_i, as shown in Figure 8. Of course, there are possibly other neighboring vertices, which will be ignored. Recall that $V_i = (a_i, b_i)$ and $V_l^i = (a_l^i, b_l^i)$, $i = 1, \ldots, 4$. Hence, there are two nonzero constants h and k such that

$$a_l^i - a_i = \frac{1}{h}(b_l^i - b_i), \quad l = 1, 3, \qquad b_l^i - b_i = \frac{1}{k}(a_l^i - a_i), \quad l = 2, 4,$$

where $hk \neq 1$.

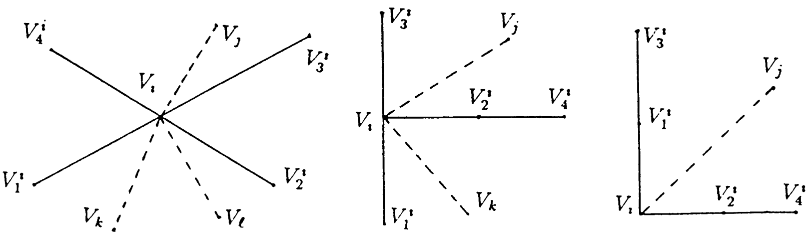

FIGURE 8

In order to solve for Z_1^i and Z_2^i in (4.3), we only require the following four equations:

$$\text{(4.9)} \quad \begin{aligned} &\frac{1}{h}Z_1^i + Z_2^i + W_1(b_1^i - b_i) = [f(V_1^i) - f(V_i)]/(b_1^i - b_i), \\ &\frac{1}{h}Z_1^i + Z_2^i + W_1(b_3^i - b_i) = [f(V_3^i) - f(V_i]/(b_3^i - b_i), \\ &Z_1^i + \frac{1}{k}Z_2^i + W_2(a_2^i - a_i) = [f(V_2^i) - f(V_i)]/(a_2^i - a_i), \\ &Z_1^i + \frac{1}{k}Z_2^i + W_2(a_4^i - a_i) = [f(V_4^i) - f(V_i)]/(a_4^i - a_i), \end{aligned}$$

where

$$W_1 = ah^{-2} + 2bh^{-1} + c, \qquad W_2 = a + 2bk^{-1} + ck^{-2}.$$

From (4.9), we now have

$$\begin{aligned} &\frac{1}{h}(b_3^i - b_1^i)Z_1^i + (b_3^i - b_1^i)Z_2^i \\ &\quad = \frac{b_3^i - b_i}{b_1^i - b_i}[f(V_1^i) - f(V_i)] - \frac{b_1^i - b_i}{b_3^i - b_i}[f(V_3^i) - f(V_i)], \\ &(a_4^i - a_2^i)Z_1^i + \frac{1}{k}(a_4^i - a_2^i)Z_2^i \\ &\quad = \frac{a_4^i - a_i}{a_2^i - a_i}f(V_2^i) - f(V_i)] - \frac{a_2^i - a_i}{a_4^i - a_i}[f(V_4^i) - f(V_i)], \end{aligned}$$

and Z_1^i and Z_2^i are given by

$$\text{(4.10)} \quad Z_1^i = \frac{1}{\delta_0}\det\begin{bmatrix} \frac{b_3^i - b_i}{b_1^i - b_i}(f(V_1^i) - f(V_i)) - \frac{b_1^i - b_i}{b_3^i - b_i}(f(V_3^i) - f(V_i)) & (b_3^i - b_1^i) \\ \frac{a_4^i - a_i}{a_2^i - a_i}(f(V_2^i) - f(V_i)) - \frac{a_2^i - a_i}{a_4^i - a_i}(f(V_4^i) - f(V_i)) & \frac{1}{k}(a_4^i - a_2^i) \end{bmatrix},$$

$$(4.11)\quad Z_2^i = \frac{1}{\delta_0}\det\begin{bmatrix} \frac{1}{h}(b_3^i - b_1^i) & \frac{b_3^i - b_i}{b_1^i - b_i}(f(V_1^i) - f(V_i)) - \frac{b_1^i - b_i}{b_3^i - b_i}(f(V_3^i) - f(V_i)) \\ (a_4^i - a_2^i) & \frac{a_4^i - a_i}{a_2^i - a_i}(f(V_2^i) - f(V_i)) - \frac{a_2^i - a_i}{a_4^i - a_i}(f(V_4^i) - f(V_i)) \end{bmatrix},$$

where

$$\delta_0 = \left(\frac{1}{hk} - 1\right)(a_4^i - a_2^i)(b_3^i - b_1^i).$$

Substituting Z_1^i and Z_2^i into (4.7), we obtain the corresponding "quasi-interpolation" formula which preserves all quadratic polynomials. Again, (4.7) can be written in the form of (4.8).

As an example, we will describe the procedure for constructing a quasi-interpolation formula of the form (4.8) for the triangulation Δ which is a (not necessarily uniform) type-1 triangulation $\Delta_{MN}^{(1)}$.To be more specific, let $\Omega = [a, b] \times [c, d]$, and

$$a = x_1 < \cdots < x_M = b, \qquad c = y_1 < \cdots < y_N = d.$$

The triangulation $\Delta_{MN}^{(1)}$ is obtained by inserting all diagonals with positive slopes to the subrectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Set $V_{ij} = (x_i, y_j)$, $f_{ij} = f(x_i, y_j)$, $i = 1, \dots, M$, $j = 1, \dots, N$. If V_{ij} is an interior vertex, then the corresponding interpolation nodes V_l^{ij}, $l = 1, 2, 3, 4$, are $V_{i,j-1}$, $V_{i+1,j}$, $V_{i,j+1}$, $V_{i-1,j}$. From (4.10) and (4.11), we have

$$(4.12)\qquad Z_1^{ij} = \frac{f_{i+1,j} - f_{ij}}{x_{i+1} - x_i} - \frac{f_{i+1,j} - f_{i-1,j}}{x_{i+1} - x_{i-1}} + \frac{f_{i,j} - f_{i-1,j}}{x_i - x_{i-1}},$$

$$(4.13)\qquad Z_i^{ij} = \frac{f_{i,j+1} - f_{ij}}{y_{j+1} - y_j} - \frac{f_{i,j+1} - f_{i,j-1}}{y_{j+1} - y_{j-1}} + \frac{f_{ij} - f_{i,j-1}}{y_j - y_{j-1}}.$$

If V_{ij} lies on the edge $x = x_1 = a$, $2 \le j \le N-1$, then V_l^{1j}, $l = 1, 2, 3, 4$, are $V_{1,j-1}$, $V_{2,j}$, $V_{1,j+1}$, and $V_{3,j}$. And from (4.10) and (4.11), we have

$$(4.14)\qquad Z_1^{1j} = \frac{f_{3j} - f_{1j}}{x_3 - x_1} - \frac{f_{3j} - f_{2j}}{x_3 - x_2} + \frac{f_{2j} - f_{1j}}{x_2 - x_1},$$

$$(4.15)\qquad Z_2^{1j} = \frac{f_{1,j+1} - f_{1,j}}{y_{j+1} - y_j} - \frac{f_{1,j+1} - f_{1,j-1}}{y_{j+1} - y_{j-1}} + \frac{f_{1,j} - f_{1,j-1}}{y_j - y_{j-1}}.$$

If V_{ij} lies on the edge $x = x_M = b$, $2 \le j \le N-1$, then V_l^{Mj}, $l = 1, 2, 3, 4$, are $V_{M,j+1}$, $V_{M-1,j}$, $V_{M,j-1}$, and $V_{M-2,j}$, and from (4.10) and (4.11), we have

$$(4.16)\qquad Z_1^{Mj} = \frac{f_{Mj} - f_{M-1,j}}{X_M - X_{M-1}} + \frac{f_{Mj} - f_{M-2,j}}{x_M - x_{M-2}} - \frac{f_{M-1,j} - f_{M-2,j}}{x_{M-1} - x_{M-2}},$$

$$(4.17)\qquad Z_2^{Mj} = \frac{f_{M,j+1} - f_{Mj}}{y_{j+1} - y_j} - \frac{f_{M,j+1} - f_{M,j-1}}{y_{j+1} - y_{j-1}} + \frac{f_{Mj} - f_{M,j-1}}{y_j - y_{j-1}}.$$

If V_{ij} lies on the edge $y = y_1 = c$, $2 \le i \le M-1$, then V_l^{i1}, $l = 1, 2, 3, 4$, are V_{i2}, $V_{i-1,1}$, V_{i3}, and $V_{i+1,1}$, and from (4.10) and (4.11), we have

$$Z_1^{i1} = \frac{f_{i+1,1} - f_{i,1}}{x_{i+1} - x_i} - \frac{f_{i+1,1} - f_{i-1,1}}{x_{i+1} - x_{i-1}} + \frac{f_{i,1} - f_{i-1,1}}{x_i - x_{i-1}}, \tag{4.18}$$

$$Z_2^{i1} = \frac{f_{i,3} - f_{i,1}}{y_3 - y_1} - \frac{f_{i,3} - f_{i,2}}{y_3 - y_2} + \frac{f_{i,2} - f_{i,1}}{y_2 - y_1}. \tag{4.19}$$

If V_{ij} lies on the edge $y = y_N = d$, $2 \le i \le M-1$, then V_l^{iN}, $l = 1, 2, 3, 4$, are $V_{i,N-1}$, $V_{i-1,N}$, $V_{i,N-2}$, and $V_{i+1,N}$, and from (4.10) and (4.11), we have

$$Z_1^{iN} = \frac{f_{i+1,N} - f_{i,N}}{x_{i+1} - x_i} - \frac{f_{i+1,N} - f_{i-1,N}}{x_{i+1} - x_{i-1}} + \frac{f_{i,N} - f_{i-1,N}}{x_i - x_{i-1}}, \tag{4.20}$$

$$Z_2^{iN} = \frac{f_{i,N} - f_{i,N-1}}{y_N - y_{N-1}} - \frac{f_{i,N-1} - f_{i,N-2}}{y_{N-1} - y_{N-2}} + \frac{f_{i,N} - f_{i,N-2}}{y_N - y_{N-2}}. \tag{4.21}$$

For V_{11}, the corresponding V_l^{11}, $l = 1, \ldots, 4$, are V_{12}, V_{21}, V_{13}, V_{31}, and from (4.10) and (4.11), we have

$$Z_1^{11} = \frac{f_{31} - f_{11}}{x_3 - x_1} - \frac{f_{31} - f_{21}}{x_3 - x_2} + \frac{f_{21} - f_{11}}{x_2 - x_1}, \tag{4.22}$$

$$Z_2^{11} = \frac{f_{13} - f_{11}}{y_3 - y_1} - \frac{f_{13} - f_{12}}{y_3 - y_2} + \frac{f_{12} - f_{11}}{y_2 - y_1}. \tag{4.23}$$

For V_{1N}, the corresponding V_l^{1N}, $l = 1, \ldots, 4$, are $V_{1,N-1}$, $V_{2,N}$, $V_{1,N-2}$, $V_{3,N}$, and from (4.10) and (4.11), we have

$$Z_1^{1N} = \frac{f_{3,N} - f_{1,N}}{x_3 - x_1} - \frac{f_{3,N} - f_{2,N}}{x_3 - x_2} + \frac{f_{2,N} - f_{1,N}}{x_2 - x_1}, \tag{4.24}$$

$$Z_2^{1N} = \frac{f_{1,N} - f_{1,N-1}}{y_N - y_{N-1}} - \frac{f_{1,N-1} - f_{1,N-2}}{y_{N-1} - y_{N-2}} + \frac{f_{1,N} - f_{1,N-2}}{y_N - y_{N-2}}. \tag{4.25}$$

For V_{M1}, the corresponding V_l^{M1}, $l = 1, \ldots, 4$, are $V_{M,2}$, $V_{M-1,1}$, $V_{M,3}$, $V_{M-2,1}$, and from (4.10) and (4.11), we have

$$Z_1^{M1} = \frac{f_{M,1} - f_{M-1,1}}{x_M - x_{M-1}} - \frac{f_{M-1,1} - f_{M-2,1}}{x_{M-1} - x_{M-2}} + \frac{f_{M1} - f_{M-2,1}}{x_M - x_{M-2}}, \tag{4.26}$$

$$Z_2^{M1} = \frac{f_{M,3} - f_{M,1}}{y_3 - y_1} - \frac{f_{M,3} - f_{M,2}}{y_3 - y_2} + \frac{f_{M,2} - f_{M,1}}{y_2 - y_1}. \tag{4.27}$$

For V_{MN}, the corresponding V_l^{MN}, $l = 1, \ldots, 4$, are $V_{M,N-1}$, $V_{M-1,N}$, $V_{M,N-2}$, $V_{M-2,N}$, and from (4.10) and (4.11), we have

$$Z_1^{MN} = \frac{f_{M,N} - f_{M-1,N}}{x_M - x_{M-1}} - \frac{f_{M-1,N} - f_{M-2,N}}{x_{M-1} - x_{M-2}} + \frac{f_{M,N} - f_{M-2,N}}{x_M - x_{M-2}}, \tag{4.28}$$

$$Z_2^{MN} = \frac{f_{M,N} - f_{M,N-1}}{y_N - y_{N-1}} - \frac{f_{M,N-1} - f_{M,N-2}}{y_{N-1} - y_{N-2}} + \frac{f_{M,N} - f_{M,N-2}}{y_N - y_{N-2}}. \tag{4.29}$$

Hence, a "quasi-interpolation" formula in $\widehat{S}_2^1(\widehat{\Delta}_{MN}^{(1)})$ which preserves all quadratic polynomials is given by

(4.30) $$S(f) = \sum_{i=1}^{M} \sum_{j=1}^{N} [f_{ij} S_{ij}^* + Z_1^{ij} T_{ij}^* + Z_2^{ij} U_{ij}^*],$$

where Z_1^{ij} and Z_2^{ij} are the values in (4.12)–(4.29). We may rewrite (4.30) in the form (4.8); that is,

(4.31) $$S(f) = \sum_{ij} f_{ij} B_{ij}^*,$$

where B_{ij}^* are the new basic functions. Hence, the approximation formula (4.31), which requires only function values at the vertices V_{ij}, provides third-order approximation. The construction of B_{ij}^* can be accomplished by using (4.12)–(4.29) in (4.30), and their explicit expressions are given in our report [2].

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